PROBLEMS RELATED TO BASES(Fourier Analysis on Arithmetical Sequences)

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PROBLEMS RELATED TO BASES

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I. Let θ and τ be two integers ≥ 2 . Define

$$B(\theta) = \text{set of real numbers normal to base } \theta$$

$$B(\tau) = \text{set of real numbers normal to base } \tau .$$

It is wellknown that

$$B(\theta) = \{ x \in R \ / \ (x\theta^n) \text{ is equidistributed (mod 1) } \}$$

and that

$$B(\theta) = B(\tau) \Leftrightarrow \frac{\log \theta}{\log \tau} \in Q .$$

Suppose now θ and τ are real numbers > 1 , not necessarily integers.

PROBLEM 1: Is it true that

$$B(\theta) = B(\tau) \Leftrightarrow \frac{\log \theta}{\log \tau} \in Q \ ?$$

The answer is surely NO! Indeed, it seems very plausible that $\sqrt{2}$ is normal to base 2: $\sqrt{2} \in B(2)$. On the other hand obviously $\sqrt{2} \notin B(\sqrt{2})$ because one term out of two in the sequence $\sqrt{2}(\sqrt{2})^n$ is 0 (mod 1) . So

$$B(2) \neq B(\sqrt{2}) \quad \text{inspite of the fact } \frac{\log 2}{\log\sqrt{2}} \in Q .$$

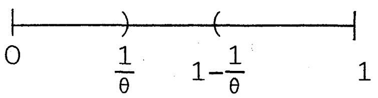
I conjecture:

If θ and τ are Pisot numbers, then

$$B(\theta) = B(\tau) \qquad \frac{\log \theta}{\log \tau} \in \mathbb{Q} .$$

(See A.Bertrand (2)).

II. Let $\theta > 2$ and consider the Cantor set $C(\theta)$



(the usual Cantor set is $C(3)$) .

Then $C(\theta)$ is the set of real numbers

$$x = (\theta - 1) \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\theta^n} , \quad \varepsilon_n = 0 \text{ or } 1 .$$

Obviously, for all <u>integer</u> $\theta \geq 3$, $B(\theta) \cap C(\theta) = \phi$.

<u>PROBLEM</u> 2: Let μ_θ be the "canonical measure" on $C(\theta)$ (I do <u>not</u> suppose that $\theta \in \mathbb{N}$) . μ_θ is the measure defined on the set of infinite 0,1-sequences. I can prove that for μ_θ-almost all $x \in C(\theta)$, x is not in $B(\theta)$. In other words

$$\mu_\theta (B(\theta) \cap C(\theta)) = 0 .$$

Is it true that $B(\theta) \cap C(\theta) = \phi$?

I can also prove that for all $x \in C(\theta)$ there exists a nonzero algebraic integer $\alpha \in \mathbb{Q}(\theta)$ such that

$$\alpha x \notin B(\theta) .$$

If this integer α was known to be rational then the conjecture

$B(\theta) \cap C(\theta) = \phi$ would be true.

On the other hand, if θ is not assumed to be a Pisot number, results seem different. I can prove that for almost all real $\theta > 1$,

$$\mu_\theta(B(\theta) \cap C(\theta)) = 1 .$$

(See my thesis (8), chapter 2).

III. Let $s_2(n)$ be the sum of the binary digits of n and $s_3(n)$ the sum of the digits of n in base 3. Straus and Senge proved (12):

$$\forall A \geq 1 \quad \forall B \geq 1 \quad \{ n \,/\, s_2(n) < A \text{ and } s_3(n) < B \}$$

is finite.

Gelfond (6) asked the 3 following questions:

QUESTION 1: Suppose $(b - 1, 2) = (b' - 1, 3) = 1$. Let a and a' be given. Show that the set

$$\{ n \in \mathbb{N} \,/\, s_2(n) \equiv a (\bmod b) \text{ and } s_3(n) \equiv a' (\bmod b') \}$$

has density equal to $1/bb'$.

QUESTION 2: Show that (p prime)

$$\{ p \leq x / \, s_2(p) \equiv a \ (\bmod b) \} \sim \frac{1}{b}\pi(x) .$$

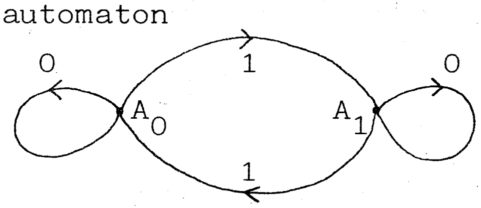
QUESTION 3: Show that

$$\{ n \in \mathbb{N} \,/\, s_2(n^2) \equiv a \ (\bmod b) \}$$

has density 1/b .

Question 1 was answered affirmatively by J.Besineau a student of mine (published in Acta Arith. (3)). Question 2 and 3 are still open. Question 2 should not be too difficult. As for Question 3, I believe it to be very deep for the following reason:

The sequence $s_2(n)$ (mod 2) is the famous Morse sequence generated by automaton



A_0 is the initial state. The automaton works as follows. Feed in the integer n written as a string of 1's and 0's . And as an output read the index i of A_i .

Because $s_2(n)$ (mod 2) is autonomous-generated, density type properties are easily established. For example

$$\text{density } \{ n \,/\, s_2(n) \equiv 0 \pmod 2 \} = 1/2 .$$

Now, J.P.Allouche (1), another student of mine proved that

$$n \longmapsto s_2(n^2) \pmod 2$$

is <u>not</u> generated by automaton. The sequence must then be quite complex.

IV. The following is a wellknown open problem. Define

$$T(n)=\begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases} .$$

To prove that $\forall n$, $\exists k$ such that $T^{(k)}(n) = 1$. It seems quite clear (??) that this problem is related to the representation of integers in both bases 2 and 3 , or what seems to amount to the same thing, to the distribution (mod 1) of $(\frac{3}{2})^n$.

So we are led to the other famous problem:

Is it true that $(\frac{3}{2})^n$ is dense (mod 1) ?

If so, then the following is obviously true. Define $\psi(x)$ as the "depth" of the rational number x :

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{\psi(x)}}}}}$$

If $(\frac{3}{2})^n$ is dense (mod 1) , then

$$\limsup_{n\to\infty} \psi\left(\left(\frac{3}{2} \right)^n \right) = +\infty .$$

G.Choquet (4) proved this in a series of Comptes Rendus a l'Acad. Sc. 1981-82 . His method was dynamical systems. Pourchet (11) had already proved that actually

$$\lim_{n\to\infty} \psi\left(\left(\frac{3}{2} \right)^n \right) = +\infty .$$

(Details are given in (14)).

Heilbronn (7), Tonkov (13), Porter (10) have proved:

$$\frac{1}{\phi(q)} \sum_{\substack{a \leq q \\ (a,q)=1}} \psi\left(\frac{a}{q} \right) \sim \frac{12}{\pi^2} \log 2 \cdot \log q \qquad (q \to \infty)$$

and J.D.Dixon (5) : for all $a < q \leq x$

$$\left| \psi\left(\frac{a}{q} \right) - \frac{12}{\pi^2} \log 2 \cdot \log q \right| < \sqrt{\log q}$$

with the exception of at most $o(x^2)$ a/q .

This leads to the following conjecture:

$$\lim_{n \to \infty} \frac{1}{n} \psi\left(\left(\frac{a}{q}\right)^n \right) = \frac{12}{\pi^2} \log 2 \cdot \log b$$

where $b = \min \{a,q\}$.

V. Back to Pisot numbers:

Let $\theta > 1$ be a <u>real</u> <u>algebraic</u> <u>number</u>. Define

$$E(x) = \{ x, x\theta, x\theta^2, x\theta^3, \ldots \}$$

and let $E'(x)$ denote the set of limit points of $E(x)$, $E''(x)$ the set of limit points of $E'(x)$,... Pisot (9) shows that if $E''(x) = \phi$ then θ is a Pisot number and $x \in Q(\theta)$.

<u>CONJECTURE</u>: If there exists a $\nu \geq 2$ such that $E^{(\nu)}(x) = \phi$, then θ is a Pisot number and $x \in Q(\theta)$.

If I knew how to prove this, I would then have another proof of Choquet's result.

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